Three important corrections

25 June 2014

Statement of Theorem 4.2.

Delete "upper (resp., lower) semicontinuous".

Statement of Theorem 4.3.

In the definition of $\mathcal{S}$ and $\hat{\mathcal{S}}$, "$v \in USC(\Omega)$" and "$w \in LSC(\Omega)$", respectively, should be replaced by "$v : \Omega \to \mathbb{R}$" and "$w : \Omega \to \mathbb{R}$".

Proof of Theorem 4.3.

We show that $u(x) := \sup_{v\in\mathcal{S}} v(x)$ is a viscosity solution. We first note $\mathcal{S} \neq \varnothing$.

It is already known that u is a viscosity subsolution. Thus, we only need to show that it is a viscosity supersolution. Suppose that it is not a viscosity supersolution. Then, there are $\phi \in C^2(\Omega)$ and $\hat{x} \in \Omega$ such that $0 = (u_* - \phi)(\hat{x}) \leq (u_* - \phi)(x)$ ($\forall x \in \Omega$), and with some $\theta > 0$,

$$F(\hat{x}, \phi(\hat{x}), D\phi(\hat{x}), D^2\phi(\hat{x})) \leq -\theta.$$

Setting $\psi(x) := \phi(x) - |x - \hat{x}|^4$, we see that

$$\begin{cases} (i) & (u_* - \psi)(\hat{x}) = 0 \leq (u_* - \psi)(x) - |x - \hat{x}|^4 \ (\forall x \in \Omega) \\ (ii) & F(\hat{x}, \psi(\hat{x}), D\psi(\hat{x}), D^2\psi(\hat{x})) \leq -\theta \end{cases} . \tag{1}$$

Thus, because of $F \in C(\Omega \times \mathbb{R} \times \mathbb{R}^n \times S^n)$ and $\phi \in C^2(\Omega)$, there is $r_0 > 0$ such that

$$F(x, \psi(x) + t, D\psi(x), D^2\psi(x)) \leq 0 \quad (\forall x \in B_{2r_0}(\hat{x}) \Subset \Omega, |t| \leq r_0). \tag{2}$$

We shall show

$$\psi(\hat{x}) < \eta(\hat{x}). \tag{3}$$

If not, since $\psi(x) \leq u_*(x) \leq \eta(x)$ ($\forall x \in \Omega$), then $\eta - \psi$ attains its minimum at $\hat{x} \in \Omega$. From the definition of η, we immediately obtain a contradiction.

Setting $\hat{\tau} := \frac{1}{3}\{\eta(\hat{x}) - \psi(\hat{x})\} > 0$, by the semicontinuity of η, we find $r_1 \in (0, r_0]$ such that

$$\eta(x) \geq \eta(\hat{x}) - \hat{\tau} \geq \psi(\hat{x}) + 2\hat{\tau} \geq \psi(x) + \hat{\tau} \quad (x \in B_{2r_1}(\hat{x})).$$

On the other hand, we may suppose

$$u(x) - \psi(x) \geq r_1^4 \quad (x \in B_{2r_1}(\hat{x}) \setminus \overline{B}_{r_1}(\hat{x})).$$

Set $\tau_0 := \min\{\hat{\tau}, \frac{r_1^4}{2}\} > 0$. We define w by

$$w(x) := \begin{cases} \max\{u(x), \psi(x) + \tau_0\} & (x \in B_{2r_1}(\hat{x})) \\ u(x) & (x \in \Omega \setminus B_{2r_1}(\hat{x})) \end{cases} .$$

Next, we shall show

$$\sup_{\Omega}(w - u) > 0. \tag{4}$$

Since $0 = (u_* - \psi)(\hat{x}) = \lim_{r \to 0} \inf\{(u - \psi)(y) \mid y \in B_r(\hat{x})\}$, we can choose $\tilde{x} \in B_{r_1}(\hat{x})$ such that $\tau_0 > (u - \psi)(\tilde{x})$.

We shall prove $w \in \mathcal{S}$. By the choice of $\tau_0, r_1 > 0$, we can show $\xi(x) \leq w(x) \leq \eta(x)$ ($\forall x \in \Omega$). Hence, we conclude (4).

Therefore, to get a contradiction, it remains to show that w is a viscosity subsolution. For $\zeta \in C^2(\Omega)$, suppose $(w^* - \zeta)(x) \leq (w^* - \zeta)(z) = 0$ ($\forall x \in \Omega$). Then, we shall verify

$$F(z, \zeta(z), D\zeta(z), D^2\zeta(z)) \leq 0. \tag{5}$$

In case of $z \in \Omega \setminus \overline{B}_{r_1}(\hat{x}) =: \Omega'$, $u^* - \psi$ takes its maximum at $z \in \Omega'$. Thus, we conclude the proof since $w = u$ in $\Omega \setminus \overline{B}_{r_1}(\hat{x})$.

Finally, in case of $z \in B_{2r_1}(\hat{x})$, $\psi + \tau_0$ is a classical subsolution, the ellipticity of F implies that it is a viscosity subsolution. Therefore, $w^* = \max\{u^*, \psi + \tau_0\}$ is again a viscosity subsolution. This fact yields a contradiction. $\square$

Proof of Theorem 5.4.

For simplicity, we write u and v for u^* and v_*, respectively. Suppsoe $\sup_{\overline{\Omega}}(u-v) =: 2\Theta > 0$. For $\alpha \in (0,1)$, setting $U(x) := u(x) - \alpha d(x)$, we see that U is a viscosity subsolution of

$$\begin{cases} F(x,U,DU,D^2U) - \tilde{\omega}_L(C_1\alpha) = 0 & \text{in } \Omega \\ \langle \boldsymbol{n}(x), DU\rangle - g(x) + \alpha = 0 & \text{on } \partial\Omega \end{cases}, \tag{6}$$

where $C_1 := \max_{\overline{\Omega}}(|Dd|+|D^2d|) > 0$. Similarly, we verify that $V(x) := v(x) + \alpha d(x)$ is a viscosity supersolution of

$$\begin{cases} F(x,V,DV,D^2V) + \tilde{\omega}_L(C_1\alpha) = 0 & \text{in } \Omega \\ \langle \boldsymbol{n}(x), DV\rangle - g(x) - \alpha = 0 & \text{on } \partial\Omega \end{cases} \tag{7}$$

For small $\alpha > 0$, we may suppose $\theta := \theta_\alpha = \sup_{\overline{\Omega}}(U-V) \geq \Theta > 0$. When $\sup_{\partial\Omega}(U-V) < \theta$, we may follow the standard argument. Thus, we shall suppose $\sup_{\partial\Omega}(U-V) = \theta$.

Let $z \in \partial\Omega$ be such that $(U-V)(z) = \theta$. For $\delta > 0$, a mapping $x \in \overline{\Omega} \to U(x) - V(x) - \delta|x-z|^2$ takes its strict local maximum at z. Then, for $\varepsilon, \delta \in (0,1)$, we set $\phi(x,y) := \frac{1}{2\varepsilon}|x-y|^2 + g(z)\langle \boldsymbol{n}(z), x-y\rangle + \delta|x-z|^2$. Let $(x_\varepsilon, y_\varepsilon) \in \overline{\Omega \cap B_s(z)} \times \overline{\Omega \cap B_s(z)}$ be a point where $\Phi(x,y) := U(x) - V(y) - \phi(x,y)$ attains its maximum over $\overline{\Omega \cap B_s(z)} \times \overline{\Omega \cap B_s(z)}$.

By $\hat{r} > 0$ for the uniform exterior sphere condition, and $\tilde{r} > 0$ for the ρ, we let $s := \frac{1}{2}\min\{\hat{r}, \tilde{r}\}$. It is easy to see

$$|U(x_\varepsilon)| \leq \max\left\{\sup_{\overline{\Omega}} U^+, \sup_{\overline{\Omega}} V^-\right\} + \sup_{\partial\Omega}|g| \times \mathrm{diam}(\Omega) =: R.$$

Since $\Phi(x_\varepsilon, y_\varepsilon) \geq \Phi(z,z)$, there is $\hat{x} \in \overline{\Omega \cap B_s(z)}$ such that $\lim_{\varepsilon\to 0}(x_\varepsilon, y_\varepsilon) = (\hat{x}, \hat{x})$. Since $\Phi(\hat{x}, \hat{x}) \geq \limsup_{\varepsilon\to 0}\Phi(x_\varepsilon, y_\varepsilon)$, we have

$$U(\hat{x}) - V(\hat{x}) - \delta|\hat{x} - z|^2 \geq \theta.$$

Hence, $\hat{x} = z$. Moreover, we have

$$\lim_{\varepsilon \to 0} \frac{|x_\varepsilon - y_\varepsilon|^2}{\varepsilon} = 0. \tag{8}$$

In view of Ishii's lemma to $U(x) - g(z)\langle \boldsymbol{n}(z), x\rangle - \delta|x - z|^2$ and $-V(y) + g(z)\langle \boldsymbol{n}(z), y\rangle$, setting $p_\varepsilon := \frac{1}{\varepsilon}(x_\varepsilon - y_\varepsilon) + g(z)\boldsymbol{n}(z)$, we can find $X, Y \in S^n$ such that

$$\begin{cases} (p_\varepsilon + 2\delta(x_\varepsilon - z), X + 2\delta I) \in \overline{J}^{2,+} U(x_\varepsilon) \\ (p_\varepsilon, -Y) \in \overline{J}^{2,-} V(y_\varepsilon) \\ \begin{pmatrix} X & 0 \\ 0 & Y \end{pmatrix} \leq \dfrac{3}{\varepsilon} \begin{pmatrix} I & -I \\ -I & I \end{pmatrix} \end{cases}.$$

When $x_\varepsilon \in \partial\Omega$, we calculate as follows:

$$\begin{aligned} \langle \boldsymbol{n}(x_\varepsilon), D_x\phi(x_\varepsilon, y_\varepsilon)\rangle &= \langle \boldsymbol{n}(x_\varepsilon), p_\varepsilon + 2\delta(x_\varepsilon - z)\rangle \\ &\geq -\tfrac{1}{2\hat{r}\varepsilon}|x_\varepsilon - y_\varepsilon|^2 + g(z)\langle \boldsymbol{n}(x_\varepsilon), \boldsymbol{n}(z)\rangle - 2\delta|x_\varepsilon - z| \end{aligned}$$

Thus, for a fixed $\alpha > 0$, for any small $\varepsilon > 0$, the continuity of g and $\boldsymbol{n}$ yields

$$\langle \boldsymbol{n}(x_\varepsilon), D_x\phi(x_\varepsilon, y_\varepsilon)\rangle - g(x_\varepsilon) \geq -\frac{\alpha}{2} > -\alpha$$

Hence, since U is a viscosity subsolution on $\partial\Omega$, we have

$$F(x_\varepsilon, U(x_\varepsilon), p_\varepsilon + 2\delta(x_\varepsilon - z), X + 2\delta I) - \tilde{\omega}_L(C_1\alpha) \leq 0.$$

In case of $x_\varepsilon \in \Omega$, the above inequality directly follows from the definition.

When $y_\varepsilon \in \partial\Omega$, we similarly have

$$\langle \boldsymbol{n}(y_\varepsilon), -D_y\phi(x_\varepsilon, y_\varepsilon)\rangle - g(y_\varepsilon) \leq \frac{\alpha}{2} < \alpha.$$

Thus, the definition of viscosity supsersolutions implies

$$F(y_\varepsilon, V(y_\varepsilon), p_\varepsilon, -Y) + \tilde{\omega}_L(C\alpha) \geq 0.$$

Therefore, we have

$$
\begin{aligned}
& \omega_0(U(x_\varepsilon) - V(y_\varepsilon)) \\
\leq\ & F(y_\varepsilon, U(x_\varepsilon), p_\varepsilon, -Y) - F(x_\varepsilon, U(x_\varepsilon), p_\varepsilon, X) + \tilde{\omega}_L(2\delta \mathrm{diam}(\Omega)) + 2\tilde{\omega}_L(C_1\alpha) \\
\leq\ & \hat{\omega}_R\left(|x_\varepsilon - y_\varepsilon|\left(|p_\varepsilon| + 1\right)\right) + \tilde{\omega}_L(2\delta \mathrm{diam}\Omega) + 2\tilde{\omega}_L(C_1\alpha)
\end{aligned}
$$

Sending $\varepsilon \to 0$, we get

$$
\omega_0(\Theta) \leq \omega_0(\theta) \leq \tilde{\omega}_L(2\delta \mathrm{diam}(\Omega)) + 2\tilde{\omega}_L(C_1\alpha).
$$

Hence, letting $\delta, \alpha \to 0$, we get $\omega_0(\Theta) \leq 0$, which yields a contradiction. □